P216 Q1 2017 Key

1) $\vec{\nabla}(\vec{A}\cdot(\vec{B}\times\vec{r})) = \vec{\nabla}(\vec{r}\cdot(\vec{A}\times\vec{B}))$ 4.5 points

let $\vec{C} = \vec{A}\times\vec{B}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{\nabla}(\vec{r}\cdot\vec{C}) = \vec{\nabla}(xC_x + yC_y + zC_z)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(xC_x + yC_y + zC_z)$$

$$= C_x\vec{i} + C_y\vec{j} + C_z\vec{k} = \vec{C} = \vec{A}\times\vec{B}$$ 8 points

2. $\vec{E} = r^{n-1}\,\vec{r}$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(r^{n-1}\vec{r}) = \vec{\nabla}(r^{n-1})\cdot\vec{r} + r^{n-1}\,\vec{\nabla}\cdot\vec{r}$$

$$\vec{\nabla}f(r) = f'(r)\,\hat{r} = f'(r)\,\frac{\vec{r}}{r}$$

$$\therefore \vec{\nabla}(r^{n-1}) = (n-1)\, r^{n-2} \cdot \frac{\vec{r}}{r}$$

$$\vec{\nabla}\cdot\vec{E} = (n-1)\, r^{n-2} \cdot \vec{r}\cdot\frac{\vec{r}}{r} + 3\, r^{n-1} \quad \text{because}$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\therefore \vec{\nabla}\cdot\vec{E} = (n+2)\, r^{n-1}$$

3.5 P

for sphere of radius R:

$$\int_{S^3} (\vec{\nabla}\cdot\vec{E})\, d^3x = \iiint (n+2)\, r^{n-1} \cdot r^2 dr \sin\theta\, d\theta\, d\varphi$$

$$= \int_0^R (n+2)\, r^{n+1} dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= R^{n+2}\,(4\pi)$$

4 P

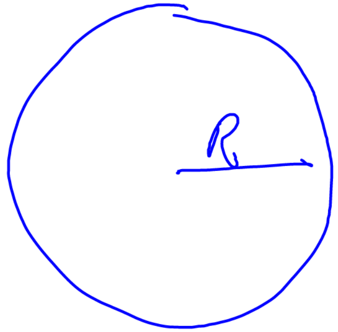
$$\oint_{\partial S^3} \vec{E}\cdot\vec{n}\, da = \iint R^{n-1} \cdot \hat{R} \cdot R^2 \sin\theta\, d\theta\, d\varphi$$

$$= R^{n+2}\,(4\pi)$$

✓ 4 P

$$d\vec{a} = R^2 \sin\theta\, d\varphi\, d\theta \cdot \hat{R}$$

3. $\vec{A} = \rho\hat{\varphi} + z\hat{z}$



on disk: $\underline{z = 0}$

$$\oint_C \vec{A}\cdot d\vec{l} = \int_0^{2\pi} R\hat{\varphi}\cdot(R\,d\varphi\,\hat{\varphi}) = R^2(2\pi) \quad 4p$$

$$\int_D (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \rho^2 & z \end{vmatrix}$$

$$= \frac{1}{\rho}\left(\hat{\rho}(0) - \rho\hat{\varphi}(0) + \hat{z}(2\rho)\right)$$

$$= 2\hat{z} \qquad 3.5$$

$$d\vec{a} = \rho\,d\rho\,d\varphi\,\hat{z} \Rightarrow (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = 2\rho\,d\rho\,d\varphi \quad 1.5p$$

$$\int_{\text{bowl}} (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_0^R 2\rho\, d\rho \int_0^{2\pi} d\varphi = R^2(2\pi)$$ ✓ 3.5p

4. $$\int_{-\infty}^{\infty} f(x)\,\delta(x^2 - x - 6)\,dx$$

delta function is even function

Let $y = |x^2 - x - 6|$, $dy = (2x-1)\,dx$

$$\int_{-\infty}^{\infty} f(x)\,\delta(y)\,\frac{dy}{|2x-1|} = \left.\frac{f(x)}{|2x-1|}\right|_{y=0}$$ 7p

$y = (x-3)(x+2) = 0$

$x = 3$ or $x = -2$

2p

$$= \frac{f(3)}{|2\times 3 - 1|} + \frac{f(-2)}{|2\times(-2) - 1|}$$

$$= \frac{f(3)}{5} + \frac{f(-2)}{5}$$ 3.5p

$$= \frac{1}{5}\left(f(3) + f(-2)\right)$$

5. $$A = \begin{pmatrix} \cos & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Eigenvalue equation : $|A-\lambda I| = \begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ \sin\theta & -\cos\theta-\lambda \end{vmatrix}$

$= -(\cos\theta-\lambda)(\cos\theta+\lambda) - \sin^2\theta = \lambda^2 - \cos^2\theta - \sin^2\theta$

$= \lambda^2 - 1 = (\lambda-1)(\lambda+1)$

?

$\therefore \lambda_1 = -1$, $\lambda_2 = 1$

Eigenvector : $\begin{pmatrix} \cos\theta+1 & \sin\theta \\ \sin\theta & -\cos\theta+1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow (\cos\theta+1)a + e^{i\varphi}\sin\theta\, b = 0$ 3.5 P

or $b = -\dfrac{1+\cos\theta}{\sin\theta}\, a = -\dfrac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, a$

normalize : $|a|^2 + |b|^2 = 1$ $= -\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\, a$

Assume a & b are real

$$|a|^2\left(1+\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right)=1 \Rightarrow a=\sin\frac{\theta}{2}$$

$$\therefore \quad b=-\cos\frac{\theta}{2}$$

$$\vec{v}_1=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$$

4.5

For $\vec{V_2}$:

$\lambda_2 = 1$

$$\begin{bmatrix} \cos\theta - 1 & \sin\theta \\ & -\cos\theta - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$(\cos\theta - 1)\,a + e^{i\varphi}\sin\theta\, b = 0$$

$$b = \frac{1-\cos\theta}{\sin\theta}\,a = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,a$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\,a$$

$$|b|^2 + |a|^2 = |a|^2\left(1 + \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right) = \frac{|a|^2}{\cos^2\frac{\theta}{2}} = 1$$

$\therefore |a| = \cos\frac{\theta}{2}$ $\qquad b = \sin\frac{\theta}{2}$

$$\therefore V_2 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad \vec{V_1} \cdot \vec{V_2} = 0$$

4.5 P

6. prove that $\int d\vec{a} \times \vec{\nabla}\phi = \oint_C \phi \, d\vec{l}$

let $\vec{A} = \vec{C}\phi$ 1.5 p.

$$\oint_C \vec{A} \cdot d\vec{l} = \vec{C} \oint_C \phi \, d\vec{l} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

$$= \int_S \vec{\nabla} \times (\vec{C}\phi) \cdot d\vec{a}$$ 5 p

$$\vec{\nabla} \times (\phi \vec{C}) = \vec{\nabla}\phi \times \vec{C}$$

$$= \int_S (\vec{\nabla}\phi \times \vec{C}) \cdot d\vec{a} = \int_S (d\vec{a} \times \vec{\nabla}\phi) \cdot \vec{C}$$ 5 p

$$\therefore \oint_C \phi \, d\vec{l} = \int_S d\vec{a} \times \vec{\nabla}\phi .$$ 1 p

7. $A = \begin{pmatrix} 1 & 2 & -i \\ 2 & 1 & i \\ i & -i & 1 \end{pmatrix}$

$A_{11} = 1, \quad A_{12} = 2, \quad A_{13} = -i$

$A_{21} = 2 \quad A_{22} = 1 \quad A_{23} = i$

$A_{31} = i \quad A_{32} = -i \quad A_{33} = 1$

$$\vec{e}'_i = \sum_{j=1}^{3} A_{ji} \vec{e}_j$$

$\vec{e}'_1 = A_{11}\vec{e}_1 + A_{21}\vec{e}_2 + A_{31}\vec{e}_3 = \vec{e}_1 + 2\vec{e}_2 + i\vec{e}_3$ 3p

$\vec{e}'_2 = A_{12}\vec{e}_1 + A_{22}\vec{e}_2 + A_{32}\vec{e}_3 = 2\vec{e}_1 + \vec{e}_2 - i\vec{e}_3$ 3p

$\vec{e}'_3 = A_{13}\vec{e}_1 + A_{23}\vec{e}_2 + A_{33}\vec{e}_3 = -i\vec{e}_1 + i\vec{e}_2 + \vec{e}_3$ 3p

$\det A = 1(1) - 2(2+1) - i(-2i - i)$

$= 1 - 6 - 3 = -8 \neq 0$ 2p

$\therefore \{\vec{e}'_i\}$ is a basis 1.5p

8. $\vec{x} = \begin{pmatrix} 1+i \\ 1-i \\ 2 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 1-i \\ 1+i \\ -2 \end{pmatrix}$

$(\vec{x}, \vec{x}) = |1+i|^2 + |1-i|^2 + 4$

$= 2 + 2 + 4 = 8$ 4 p

$(\vec{y}, \vec{y}) = |1-i|^2 + |1+i|^2 + 4 = 8$ 4 p

$(\vec{x}, \vec{y}) = (1+i)^* (1-i) + (1-i)^* (1+i) - 4$

$= 1 - 1 - 2i + 1 - 1 + 2i - 4 = -4$ 4 p

$(\vec{y}, \vec{x}) = (\vec{x}, \vec{y})^* = -4$ 0.5 p

9. If A and B are 2 commuting operators

and $\vec{x}$ is an eigenvector of A

$A\vec{x} = a\vec{x}$, $BA\vec{x} = B(a\vec{x}) = aB\vec{x}$

But $AB\vec{x} = A(B\vec{x}) = BA\vec{x} = B(a\vec{x}) = a(B\vec{x})$

$\therefore$ $B\vec{x}$ is an eigenvector of A with some eigenvalue a, and thus $B\vec{x}$ must be proportional to $\vec{x}$

$\therefore$ $B\vec{x} = b\vec{x}$ 7 p

If $A\vec{x} = a\vec{x}$, $B\vec{x} = b\vec{x}$ where $\vec{x}$ is a common eigenvalue:

$AB\vec{x} = A(b\vec{x}) = b(A\vec{x}) = ba\vec{x}$

$BA\vec{x} = B(a\vec{x}) = a(B\vec{x}) = ab\vec{x}$

$\therefore$ $[A,B]\vec{x} = 0$ $\Longrightarrow$ $[A,B] = 0$ 5.5 p

10. $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$

$\det A = 1(40-16) - 2(16-12) - 3(-8+15) = 24 - 8 - 21 = -5$

$\mathrm{Cof}(A)_{11} = (5\times 8 - 16) = 24$ 1p

$(\mathrm{Cof}\,A)_{21} = -(16-12) = -4$ 1p

$(\mathrm{Cof}\,A)_{12} = -(2\times 8 - 12) = -4$ 1p

$(\mathrm{Cof}\,A)_{22} = 8 - 9 = -1$ 1p

$(\mathrm{Cof}\,A)_{13} = (-8+15) = 7$ 1p

$(\mathrm{Cof}\,A)_{23} = -(-4+6) = -2$ 1p

$(\mathrm{Cof}\,A)_{31} = (-8+15) = 7$ 1p

$(\mathrm{Cof}\,A)_{32} = -(-4+6) = -2$ 1p

$(\mathrm{Cof}\,A)_{33} = (5-4) = 1$ 1p

$\therefore \mathrm{Cof}\,A = \begin{pmatrix} 24 & -4 & 7 \\ -4 & -1 & -2 \\ 7 & -2 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A}(\mathrm{Cof}\,A)^T = -\frac{1}{5}\begin{pmatrix} 24 & -4 & 7 \\ -4 & -1 & -2 \\ 7 & -2 & 1 \end{pmatrix}$

3.5p